A NUMERICAL RELATIVITY APPROACH TO BRANEWORLD COSMOLOGY

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Abstract

Due to the non-linearity and the complexity, it is not very realistic to study the full dynamics in closed form, especially the behaviour at high energy regime where the fields are so strong that perturbation methods do not apply. Therefore, it is simply here needed to use a numerical approach. A simple finite difference method has been utilized to solve non-linear wave equation in flat 3+1 dimension space time under axisymmetry with source term f in Braneworld. Interesting results have been obtained and they are visualized in 2-D and 3-D.

Keywords: 3+1 dimension spacetime, axisymmetry, braneworld.

Introduction

The 3+1 formalism is an approach to general relativity and to Einstein equations that relies on the slicing of the four dimensional spacetime by three-dimensional surfaces. From the mathematical point of view, this procedure allows to formulate the problem of resolution of Einstein equations as a constraints. The 3+1 formalism should not be confused with the 1+3 formalism, where the basis structure is a congruence of one dimensional curves instead of three-dimensional surfaces.

Today, most numerical codes for solving Einstein equations are based on the 3+1 formalism We will present the evolution schemes of the 3+1 Einstein equations, putting some emphasis on the most successful scheme to date.

Evolution Schemes

There exist various formalisms of GR, among which only the ones that are strongly hyperboliccan be used as a well-defined formalism of an initial value problem.

The generalized harmonic (GH) formalism used the gauge source functions

$$H^{\alpha} \simeq \nabla^{\beta}\nabla_{\beta}x^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}g^{\mu\nu} \equiv -\Gamma^{\alpha} \qquad (1)$$

as fundamental variables. The notation$\simeq$ means the equation is a constraint relation. Einstein's equations can now be written as

$$-\frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta} - g^{\alpha\beta}_{(\ ,\mu}g_{\nu)\beta,\alpha} - H_{(\mu,\nu)} + H_{\beta}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\mu\alpha} = k_d\left(T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T\right) \qquad (2)$$

A coordinate gauge choice can now be realized via specifying the H_{μ}'s. As long asH_{μ} does not include derivative of metric functions., the principle part of the above equation$-\frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta}$ is manifestly strongly hyperbolic.

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Both the generalized harmonic formalism and the BSSN formalism are widely used in the literature, yet none of them is sufficient to simulate braneworld and we have to develop them further. In this research, the generalized harmonic formalism will be employed to evolve the braneworld spacetime(Bona, 1995).

Numerical Methods

The equations of motion of gravitational theory are non-linearly coupled partial differential equations (PDEs). Due to the non-linearity and the complexity, it is not very realistic to study the full dynamics in closed form, especially the behavior at high energy regime where the fields are so strong that perturbation methods do not apply. We are use a numerical approach. In this section we introduce finite difference approximation (FDA) methods to solve the PDEs. The focus is in the various tests to distinguished numerical solutions from numerical artifacts (Bardeen,J.M., 1970).

Finite Difference Approximation

To demonstrate the concepts in a less abstract way, let us consider the following model problem, which is non-linear equation in flat 3+1 dimension spacetime under axisymmetry with source term *f* (which does not depend on the eave function Φ). This model problem includes a few features that are important for numerical calculation in braneworlds. The equation is assumed to be $\left(-\partial_{tt}+\partial_{xx}+\partial_{yy}+\partial_{zz}\right)\Phi+\Phi^2=f$ in Cartesian coordinates, or

$$\left(-\partial_{tt}+\partial_{\rho\rho}+\frac{1}{\rho}\partial_{\rho}+\partial_{zz}\right)\Phi+\Phi^2=f \qquad (3)$$

in cylindrical coordinates (t, ρ, ϕ, z) that are adapted to the axisymmetry. Therefore the axisymmetry implies $\partial_{\phi}\Phi=0$, which has been applied in (3). Let us assumed the spatial domain is $\rho\,\epsilon[0,\rho_{max}], z\,\epsilon[0,z_{max}]$.

The whole domain, both spatial and temporal, is divided into discrete grids (or meshes). In principle this division can be arbitrary, as long as the grid/mesh elements are small. To be more specific and to honor simplicity, here let us employ uniform grid. Therefore the spatial domain can be

$$\rho_i=(i-1)\Delta\rho\,;i=1,2,\dots,n_{\rho}\quad where\ \ \Delta\rho=\frac{\rho_{max}}{n_{\rho}-1}\,; \qquad (4)$$

$$z_j=(j-1)\Delta z\,;j=1,2,\dots,n_{z}\quad where\ \ \Delta z=\frac{z_{max}}{n_{z}-1}\,; \qquad (5)$$

For simplicity let us choose $\Delta\rho=\Delta z=h$. The time domain is also discreted and the time interval between two subsequent discretized time levels can be expressed as Δt. $\Delta t/h$is called the Courant factor.

One uses notation

$$\Phi^n_{i,j}\equiv\Phi\left(t^n,\rho_i,z_j\right)\equiv\Phi\big((n-1)\Delta t,(i-1)\Delta\rho,(j-1)\Delta z\big), \qquad (6)$$

and similar notation for function *f*. We replace the differential operators by their FDA operators with second order accuracy:

$$\partial_{\rho\rho}\Phi\rightarrow\frac{\Phi^n_{i+1,j}-2\Phi^n_{i,j}+\Phi^n_{i-1,j}}{h^2}, \qquad (7a)$$

$$\partial_\rho \Phi \to \frac{\Phi^n_{i+1,j}-\Phi^n_{i-1,j}}{2h}, \tag{7b}$$

$$\partial_{zz} \Phi \to \frac{\Phi^n_{i,j+1}-2\Phi^n_{i,j}+\Phi^n_{i,j-1}}{h^2}, \tag{7c}$$

$$\partial_{tt} \Phi \to \frac{\Phi^{n+1}_{i,j}-2\Phi^n_{i,j}+\Phi^{n-1}_{i,j}}{(\lambda h)^2}. \tag{7d}$$

The FDA operators are obtained by Taylor expansions such as

$$\Phi^n_{i+1,j} = \Phi^n_{i,j} + h\Phi_{,\rho} + \frac{h^2}{2!}\Phi_{,\rho\rho} + \frac{h^3}{3!}\Phi_{,\rho\rho\rho} + \frac{h^4}{4!}\Phi_{,\rho\rho\rho\rho} + O(h^6),$$

which yield

$$\frac{\Phi^n_{i+1,j}-2\Phi^n_{i,j}+\Phi^n_{i-1,j}}{h^2} = \partial_{\rho\rho}\Phi + \frac{h^2}{12}\Phi_{,\rho\rho\rho\rho} + O(h^4), \tag{8}$$

The term $\frac{h^2}{12}\Phi_{,\rho\rho\rho\rho} + O(h^4) = O(h^2)$ is the difference between the exact operator and the FDA operator, which is called truncation error. When h is small (so that the truncation error in not significant), the differential operators can be replaced by their FDA counter parts. Other FDA operators in (7) can be obtained similarly. The discretized PDE reads

$$\frac{\Phi^{n+1}_{i,j} - 2\Phi^n_{i,j} + \Phi^{n-1}_{i,j}}{(\lambda h)^2} + \frac{\Phi^n_{i,j+1} - 2\Phi^n_{i,j} + \Phi^n_{i,j-1}}{h^2} + \frac{1}{\rho_i}\frac{\Phi^n_{i+1,j} - \Phi^n_{i-1,j}}{2h}$$
$$+ \frac{\Phi^n_{i,j+1}-2\Phi^n_{i,j}+\Phi^n_{i,j-1}}{h^2} + \left(\Phi^n_{i,j}\right)^2 = f^n_{i,j} \tag{9}$$

Now we are ready to introduce the general notations to make the discussion clearer. A set of PDFs, such as equation (3), can be collectively denoted as

$$Lu = f \tag{10}$$

where L stands for differential operators and all other operations, u stands for the fundamental variables (the unknown functions) to solve for, and f stands for terms in the equations that do not include u,. In equation (3), $u = \Phi$ and$u = L\Phi = \left(-\partial_{tt} + \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}\right)\Phi + \Phi^2$.

The discrete FDA operators, such as equation (8) , can be collectively denoted as

$$A\Phi = \varepsilon\, \Phi + h^p . E\Phi\,, \tag{11}$$

where A stands for the FDA version of the exact operatorε. h^pmeans that the approximation level is of p-th order in h, E stands for the error operator — more specifically,$h^p . E\Phi$ is the error. Using (11), we can discretize (10) as

$$L^h u^h = f^h \tag{12}$$

where h is to label resolution. An example of (12) is (9).

In (9), the approximation is of second order in h. Generally the approximation order of L^h is p, which can be formally expressed as

$$L^h = L + h^p E \tag{13}$$

From the discussion above, one see that the validity of FDA needs to be built upon the following two assumptions: (1) the function Φ is smooth; (2) h is small, so that the truncation error is not significant.

However, these two conditions are not sufficient to guarantee that the numerical resultu^h is actually a approximation of the exact solution u. Therefore systematic test mechanisms need to be developed to distinguish numerical solutions from numerical artifacts (Baumgarte, T. W., and Shapiro, S. L., 2003).

Tests

First, often it is neither practical nor necessary to let equation (12) be satisfied exactly. Instead, (12) is considered to be satisfied when residual$r^h \equiv L^h u^h - f^h$is sufficiently small. Again, "small" does not have any measurable meaning yet.

Multiplying equation (9) by ρ_i, we get the following equation

$$-\frac{\rho_i\left(\Phi_{i,j}^{n+1} - 2\Phi_{i,j}^{n} + \Phi_{i,j}^{n-1}\right)}{(\lambda h)^2} + \frac{\rho_i\left(\Phi_{i,j+1}^{n} - 2\Phi_{i,j}^{n} + \Phi_{i,j-1}^{n}\right)}{h^2} + \frac{\Phi_{i+1,j}^{n} - \Phi_{i-1,j}^{n}}{2h}$$

$$+\frac{\rho_i\left(\Phi_{i,j+1}^{n} - 2\Phi_{i,j}^{n} + \Phi_{i,j-1}^{n}\right)}{h^2} + \rho_i\left(\Phi_{i,j}^{n}\right)^2 = \rho_i f_{i,j}^{n}\,. \qquad (14)$$

(9) and (14) share exactly the same numerical properties, such as convergence, smoothness, regularly, etc. But the two residuals have different numerical values. Therefore, the residual being "small", has no absolute meaning.

This feature can be expressed in a more abstract way as: $Lu = f$and$g.Lu = g.f$ have the same numerical properties. Here g is a non-zero smooth function over the domain. For example g can be an arbitrary non-zero constant to make the residual take any value. Therefore, the absolute value of residual does not have any meaning. So, how to distinguish between a numerical solution and a numerical artifact? And how small is the residual to be considered sufficiently small? These questions will be answered by the following analysis.

Assume the numerical resultu^h that satisfies$L^h u^h = f^h + r^h$ is obtained, wherer^h is the residual. Generally,u^h is a numerical solution, if the following equation is satisfied whenu^h is substituted back into equation (10)

$$\lim_{h\to 0} Lu^h - f^h = 0\,. \qquad (15)$$

Let us see what it means

$$Lu^h - f^h = L^h u^h - h^p E u^h - f^h = r^h - h^p E u^h = r^h + O(h^p). \qquad (16)$$

Therefore (15) is satisfied, ifr^h is negligible compared to $h^p E u^h$ (in the sense r^h is small).

However technically it is impossible to apply a continuous operation L to discrete functionu^h, and then (15) can only be understood formally. Instead,u^h is considered a numerical solution, if

$$\lim_{h\to 0} r_I^h = 0 \qquad (17)$$

where$r_I^h \equiv L_I^h u^h - f^h$, where $L_I^h \neq L^h$ that satisfies $\lim_{h\to 0} L_I^h = L$.

SinceL_I^h is independent ofL^h (a different discretization),r_I^h is called independent residual.

For the model problem, we can use following discretization as the independent discretized operators

$$\partial_{rr}\Phi \rightarrow \frac{2\Phi^n_{i,j}-5\Phi^n_{i+1,j}+4\Phi^n_{i+2j}+\Phi^n_{i+3j}}{h^2}, \tag{18a}$$

$$\partial_{r}\Phi \rightarrow \frac{3\Phi^n_{i,j}-4\Phi^n_{i+1,j}+\Phi^n_{i+2,j}}{2h}, \tag{18b}$$

$$\partial_{zz}\Phi \rightarrow \frac{2\Phi^n_{i,j}-5\Phi^n_{i,j+1}+4\Phi^n_{i,j+2}-\Phi^n_{i,j+3}}{h^2}, \tag{18c}$$

$$\partial_{tt}\Phi \rightarrow \frac{2\Phi^n_{i,j}-5\Phi^{n-1}_{i,j}+4\Phi^{n-2}_{i,j}-\Phi^{n-3}_{i,j}}{(\lambda h)^2}. \tag{18d}$$

This discretization is different from (7) and is also of the second order accuracy.

In general, the approximation order ofL^h_I is denoted as *m*, therefore

$$L^h_I = L + h^m E_I = L^h - h^p E + h^m E_I \tag{19}$$

$$r^h_I = L^h_I u^h - f^h = (L^h - h^p E + h^m E_I)u^h - f^h = r^h - h^p E u^h + h^m E_I u^h . \tag{20}$$

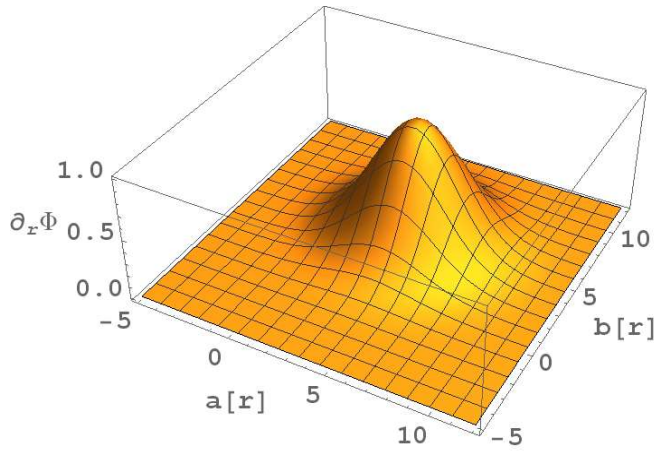

Figure 1 3D Profile of $\partial_r\Phi$ in terms of a(r) and b(r).

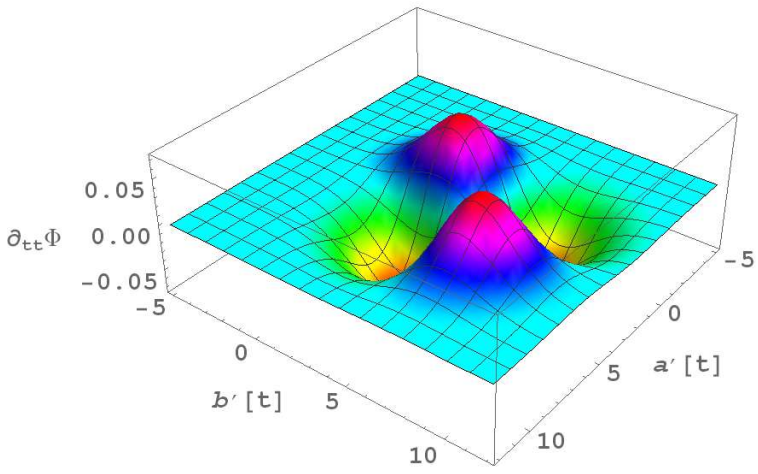

Figure 2 3D Profile of $\partial_{tt}\Phi$ in terms of a′(t)and b′(t).

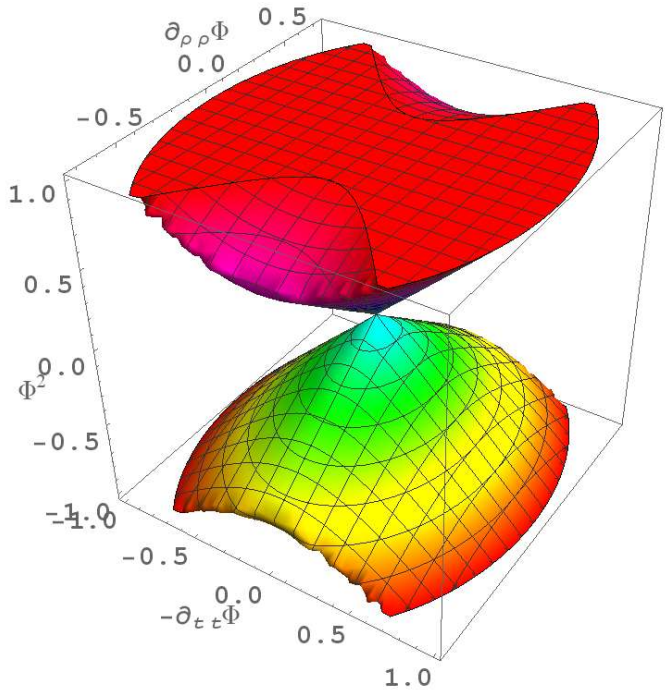

Figure 3 3D Profile of possible region distribution form main contributor in brane world scenario.

Again, here it is required that $\|r^h\|$ is negligible compared to min($\|h^p E u^h\|, \|h^m E_I u^h\|$) , therefore the independent residualr_I^h converges to zero at min*(p, m)-th* order. Here$\|u\|$ is the form of u.

For the model problem,$p = m = 2$, therefore the independent residual behaves as a second order quantity: when h decreases to *h/2* , the independent residualr_I^h decreases to $r_I^{(h/2)} = \frac{1}{4} r_I^h$.

Tests for General Relativity

For a numerical problem, often there are a certain number of equations to solve, for an equal number of fundamental variables (the unknown functions). If the number of equations is less than the number of unknown functions, in principle there are no unique solutions. On the other hand, in GR, the number of equations is greater than the number of unknown functions. In this case the redundant equations are called constraints.

As an example, in 3+1 formalism of GR, there are six functionsγ_{ij} to be solved for, by solving the six evolutionary equations. The other four equations are the Hamiltonian constraint and momentum constraints. Analytically, if the constraints are satisfied initially the consistency guarantees them to be satisfied at all times, as long as the evolutionary equations are satisfied during the evolution. However, numerically there are always small violations to the constraints, and there is no guarantee the violations are controllable. Therefore, for general relativity, the constraints need to be tested as well, i.e., in order to make sure all the components of Einstein's equations are satisfied, both the independent residual test and the convergence test for constraint are needed.

Equivalently, in the case a certain formalism of GR is employed to obtain the numerical results, the results can be substituted into another formalism of GR to produce residuals, and the residuals should converge at the expected order. For example, one can use generalized harmonic formalism to obtain the solution, and then substitute the solution into original Einstein's equations to get residuals, and check whether the residuals converge as expected (Baumgarte, T.W., 2007).

Concluding Remarks

In this paper, a numerical relativity approach to Braneworld cosmology have been presented using mathematica coding to solve non-linear wave equation in flat 3+1 dimension spacetime under axisymmetry with source term *f* in Braneworld. It has been attempted to make use of finite different analysis and they are visualized. .The spatial contribution in the brane world spacetimes gives formal Gaussian distribution and second-order-time derivative gives dip and rise spacetimes structure as expected before, Another interesting feature of the braneworld might be closely connected to the usual worm hole structure if one is to ignore the rest two spatial dimensions to compare with other three including branewold dimension. It is visualized in Figure 3.

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